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Editorial

BANACH LIMITS CORRESPONDING TO RESIDUES OF THE ζ-FUNCTION

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1. Introduction and Preliminaries

In last 30 years singular traces have proved to be a useful tool in different areas of mathematics and its physical applications [2, 3]. Usually the singular traces of operators arising in geometrically or physically inspired problems are computed using the ζ-function residue or a heat kernel asymptotic expansion of these operators (e.g. see [1, 9]). In this paper we study residues of the ζ-function from the classical harmonic analysis point of view. These results extend and complement those in [8].

Throughout the paper $L_\infty(0,\infty)$ denotes the space of all (equivalence classes of) real-valued essentially bounded Lebesgue measurable functions on $(0,\infty)$ equipped with the essential supremum norm.

A linear functional γ on $L_\infty(0,\infty)$ is called an extended limit if it is a Hahn-Banach extension of the ordinary limit (at ∞). For an operator A from the weak trace class ideal the extended residue of the ζ-function is defined as a linear extension of the following weight:

$$\zeta_\gamma(A) := \gamma\left(t \ \mapsto \frac{1}{t} Tr\left(A^{1+\frac{1}{t}}\right)\right), A \geq 0.$$

A linear functional B on the space of all bounded sequences l_∞ is called a Banach limit if it is positive, normalized and translation invariant.

2. Main Results

It follows from results of [4, 5] that there is a canonical bijection between positive normalized singular traces on the weak trace class ideal and Banach limits.

Since extended ζ-function residues form a subclass in the class of all positive normalised singular traces [9], they correspond to Banach limits of a special type.

It was proved in [8] that if γ is an extended limit on $L_\infty(0,\infty)$, then an extended zeta-function residue ζ_γ corresponds to a Banach limit defined by setting

$$(1) \qquad B_{\gamma,1}(x) = \log 2 \cdot \gamma\left(t \longmapsto \frac{1}{t} \sum_{k=0}^{\infty} x_k 2^{-k/t}\right), x \in l_\infty.$$

Some characteristics of Banach limits of the form (1) were given in [8]. In this note we further characterize these Banach limits.

Lemma 2.1. If γ is an extended limit on $L_\infty(0,\infty)$, then for every *n* = 1, 2, …. the functional $B_{\gamma,n}$ is defined on l_∞ by setting

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$$B_{\gamma,n}(x) = \frac{\log^n 2}{(n-1)!} \cdot \gamma\left(t \longmapsto \frac{1}{t^n} \sum_{k=0}^{\infty} (k+1)^{n-1} x_k 2^{-k/t}\right) , x \in l_\infty.$$

is a Banach limit.

There is a relation between all Banach limits $B_{\gamma,n}$ from Lemma 2.1. To describe it we define generalized Cesaro operators $C_n: l_\infty \to l_\infty$ as follows:

$$(C_n x)_m = \frac{n}{(m+1)^n} \sum_{k=0}^{m} (k+1)^{n-1} x_k, n = 1, 2, \dots$$

Note that when n = 1 we obtain the classical Cesaro operator.

Lemma 2.2. If γ is an extended limit on $L_\infty(0,\infty)$ and $B_{\gamma,n}$ are defined as in Lemma 2.1, then one has

$$B_{\gamma,n} = B_{\gamma,n+1} \circ C_n, \forall\, n \geq 1.$$

The previous two results are proved in [8] for n = 1 and 2 only.

The operators C_n have interesting properties. In particular, a Banach limit B is Cesaro invariant (that is, $B = B \circ C$) if and only if it is invariant with respect to all C_n (that is, $B = B \circ C_n$ for every $n = 2, 3, \dots$).

This fact implies the following result:

Proposition 2.3. Let γ be an extended limit on $L_\infty(0,\infty)$ and $B_{\gamma,n}$ are defined as in Lemma 2.1. If some $B_{\gamma,n}$ is Cesaro invariant, then all $B_{\gamma,i}$, $1 \leq i \leq n$ are Cesaro invariant.

The following result is an analogue of the classical Sucheston result [7] for Banach limits corresponding to ζ- function residues. Also this should be compared with [6, Corollary 13].

Theorem 2.4. If γ is an extended limit on $L_\infty(0,\infty)$, then the Banach limit $B_{\gamma,1}$ satisfies the condition $-p(-x) \leq B_{\gamma,1} \leq p(x)$, $x \in l_\infty$ where the convex functional p on l_∞ is defined by the setting

$$p(x) \coloneqq \lim_{m\to\infty} \limsup_{n\to\infty} \frac{m}{(n+1)^m} \sum_{k=0}^{n} (n+1-k)^{m-1} x_k, x \in l_\infty.$$

The following result shows that $B_{\gamma,1}$ cannot be an extreme point of the set of all Banach limits.

Theorem 2.5. The set of Banach limits corresponding to the residues of ζ-function is disjoint with the set of extreme points of the set of all Banach limits.

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